NOTIONS OF ABSOLUTELY CONTINUOUS SUBSPACE FOR NONSELFADJOINT OPERATORS

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Abstract. We give an example of an operator with different weak and strong absolutely continuous subspaces, and a counterexample to the duality problem for the spectral components. Both examples are optimal in the scale of compact operators.
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Introduction. The known definitions of the absolutely continuous (a. c.) subspace for nonselfadjoint operators fall into two groups: the weak ones and the strong one. The strong definition was first suggested by L. Sakhnovich [1] in the case of dissipative operators.

Definition 1. *Let L be a completely non-selfadjoint dissipative operator in a Hilbert space H with a bounded imaginary part V. Any of the following coinciding subspaces is called the strong a. c. subspace H_{ac} of L,*

(*i*) *The invariant subspace of L corresponding to the canonical factorization of its characteristic function;*

(*ii*) *The minimal subspace containing all the invariant subspaces X of L such that $L|_X = WAW^{-1}$ for an a. c. selfadjoint operator A and a bounded and boundedly invertible operator $W: X \to X$;*

(*iii*) $Clos \left\{ u \in H: \quad V^{1/2} (L-z)^{-1} u \big|_{\mathbb{C}_+} \in \mathbf{H}^2_+ \right\}.$

The notation we use is given in the end of Introduction. The correspondence meant in definition (i) is the one between invariant subspaces of an operator and regular factorizations of its characteristic function in the framework of the Szökefalvi - Nagy - Foiaş functional model. The equivalence of (ii) and (iii) is a corollary of this correspondence. The equivalent definition (iii) in model-free terms was found in [2]. In this latter form, the definition was generalized to non-dissipative perturbations of selfadjoint operators [2] and non-contractive perturbations of unitary ones [3].

The weak definitions of the a. c. subspace are obtained if we try to generalize directly the "selfadjoint" definition using the property expressed by the Riesz brothers theorem as a substitute for the absolute continuity of (non-existent, in general) spectral measure. This leads to spaces defined by the requirement that the matrix element of the resolvent be of the Hardy class H^p. They were first introduced and studied in [4, 5] and are called weak a. c. subspaces. A weak a. c. subspace contains the strong one because the

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restriction of the operator to the a. c. subspace is quasi-similar to an a. c. selfadjoint operator. A natural question is whether these subspaces coincide. So far, it was answered in affirmative in two situations ($p = 2$):

(i) When L is dissipative [6];

(ii) When the characteristic function of L has weak boundary values a. e. on the real axis [7]. This holds true, for instance, for trace class perturbations of a selfadjoint operator, and, more generally, if the function admits scalar multiple.

The first main result of the present paper given by theorems 3 and 4 is —

- An example of an operator with different weak and strong a. c. subspaces.

The example in theorem 4 is a (non-dissipative) perturbation of a selfadjoint operator. Theorem 3 provides an analogous result for perturbations of unitary operators. The operators we construct are in fact similar to the standard bilateral shift [8] in the unit circle case and to the generator of bilateral shift on $\mathbb{R}$ in the real line case.

Whichever definition of the a. c. subspace is used, it is natural to ask whether the orthogonal complement of it coincides with the singular subspace of the adjoint operator. The latter is defined to be the closure of the set of vectors such that the matrix element of the resolvent on such a vector has zero jump a. e. while crossing the essential spectrum (real line and unit circle in the case of perturbations of selfadjoint and unitary operators, respectively). This question is sometimes referred to as the duality problem for spectral components and is known to have affirmative answer in the case of trace class perturbations [3, 4]. Our second main result is theorem 5 —

- An example of an operator with trivial singular subspace of the adjoint operator and nontrivial orthogonal complement of the weak a. c. subspace.

The example can be described as a non-contractive conjunction of two bilateral shifts.

The examples in theorems 3 and 5 are optimal in the sense that they are additive perturbations of unitary operators by an operator arbitrarily close to the trace class in the sense that its s-numbers can be chosen to be estimated above by an arbitrary given monotone non-summable sequence.

Theorems 3 and 4 are proved in §1, theorem 5 — in §2. In §3 we analyze the weak definition of the a. c. subspace for $p \neq 2$, and show that in the dissipative case it gives the same subspace as for $p = 2$.

NOTATION:

$\mathbb{C}_\pm = \{z : \pm \mathrm{Im}\, z > 0\}$, $\mathbb{T} = \{z \colon |z| = 1\}$; $\mathbb{D} = \{z \colon |z| < 1\}$.

$H^p_\pm$, $0 < p \leq 2$, - Hardy classes of analytic functions in $\mathbb{C}_\pm$, respectively.

H^2 - the Hardy space for the unit disk.

For functions with values in a Hilbert space H,

$\mathbf{H}^2_\pm$ - the Hardy space of H-valued functions in $\mathbb{C}_\pm$, respectively. The norm of an $f \in \mathbf{H}^2_\pm$ is given by $\sup_{\varepsilon>0} \int_{\mathbb{R}} \|f(k \pm i\varepsilon)\|^2_H \, dk < \infty$;

$\mathbf{H}^2$ - the Hardy space of H-valued functions in the unit disk.
$\{e_n\}$ - the standard basis in $l^2(\mathbb{Z})$.
For an operator L in a Hilbert space,
$\mathcal{D}(L)$ - the domain of L;
$f_{u,v}(z) = \left\langle (L-z)^{-1} u, v \right\rangle$.
$\mathbf{S}^p$, $p \geq 1$, - the Shatten - von Neumann classes of compact operators with summable p-th power of their singular numbers.

The subscripts $_\pm$ with functions in the complex plane stand for their respective restrictions to $\mathbb{C}_\pm$.

Various subspaces corresponding to abstract operators are defined in the paper. We will often suppress the explicit indication of the operator in the notation for the subspaces when it is clear which operator it refers to.

§1. Let L be a closed operator in a Hilbert space H such that $\sigma(L) \cap \mathbb{C}_\pm$ are discrete sets. For any $p \leq 2$ one can define the following subspaces in H,

$$H_{ac}^{w,p}(L) \stackrel{def}{=} Clos\, \widetilde{H_{ac}^{w,p}}(L), \tag{1}$$

$$\widetilde{H_{ac}^{w,p}}(L) \stackrel{def}{=} \left\{ \begin{array}{ll} u \in H: & (L-z)^{-1} u \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\ & \left\langle (L-z)^{-1} u, v \right\rangle_\pm \in H_\pm^p \text{ for all } v \in H. \end{array} \right\}.$$

In the case $p = 2$ this subspace is the *weak a. c. subspace* of the operator L. Elements of the linear set $\widetilde{H_{ac}^{w}}(L)$ are called *weak smooth vectors*. If L is self-adjoint, then for all $p, 1 < p \leq 2$, the subspaces $H_{ac}^{w,p}(L)$ coincide with the a. c. subspace of the operator L defined in the standard way. We include a proof of this folklore-type assertion in the Appendix. We shall omit the index p in our notation in the case $p = 2$ and write H_{ac}^w for $H_{ac}^{w,2}$. It follows from the uniform boundedness principle that for $u \in \widetilde{H_{ac}^{w,p}}$ the H^p-norms of the functions $f_{u,v}(z)$ are bounded above when v ranges over the unit ball in H.

For clarity, we restrict our consideration to the situation of the perturbation theory. From now on, it is assumed that

(A) *L is a completely nonself-adjoint operator of the form $L = A + iV$, $A = A^*$, $V = V^*$, $\mathcal{D}(L) := \mathcal{D}(A) \subset \mathcal{D}(V)$, and V is A-bounded with a relative bound less than 1, that is, $\|Vu\|^2 \leq a\|Au\|^2 + b\|u\|^2$, $a < 1$, for all $u \in \mathcal{D}(A)$.*

This assumption implies, in particular, that

$$i\tau\,(L + i\tau)^{-1} \xrightarrow{s} I,\ \tau \to \pm\infty, \tag{2}$$

and that the operator $|V|^{1/2}(L-z)^{-1}$ is defined as a bounded operator for all $z \in \rho(L)$.

Definition 2 ([2])**.** *The subspace*

$$H_{ac}(L) \stackrel{def}{=} Clos\, \widetilde{H_{ac}}(L),$$

$$\widetilde{H_{ac}}(L) \stackrel{def}{=} \left\{ \begin{array}{ll} u \in H\colon & (L-z)^{-1} u \text{ is analytic in } \mathbb{C}\setminus\mathbb{R}, \\ & \left(|V|^{1/2} (L-z)^{-1} u\right)_{\pm} \in \mathbf{H}^2_{\pm} \end{array} \right\}.$$

is called the strong a. c. subspace of the operator L. Elements of the set $\widetilde{H_{ac}}(L)$ are called strong smooth vectors.

Notice that there exists a natural generalization of this definition applicable to operators which do not satisfy the assumption (A) [7].

The main property of the strong smooth vectors is expressed by the following

Proposition [2, Theorem 4]**.** *There exists a Hilbert space $\mathcal{N}$, an a. c. self-adjoint operator A_0 in $\mathcal{N}$, and a bounded operator $P\colon \mathcal{N} \to H$ such that $P\mathcal{N} = \widetilde{H_{ac}}(L)$ and the equality*

$$(L-z)^{-1} Pg = P\,(A_0 - z)^{-1} g.$$

holds for all $g \in \mathcal{N}$ and $z \notin \mathbb{R}$, $z \in \rho(L)$.

Corollary 1. *$H^w_{ac}(L) \supset H_{ac}(L)$.*

A similar theory is available for perturbations of unitary operators [3]. Let T be a bounded operator such that $\sigma(T)$ has no accumulation points off $\mathbb{T}$.

Definition 3. *The weak a. c. subspace of the operator T is the set*

$$H^w_{ac}(T) \stackrel{def}{=} Clos\, \widetilde{H^w_{ac}}(T),$$

$$\widetilde{H^w_{ac}}(T) \stackrel{def}{=} \widetilde{H^w_{+}}(T) \cap \widetilde{H^w_{-}}(T),$$

$$\widetilde{H^w_{+}}(T) \stackrel{def}{=} \left\{ \begin{array}{ll} u \in H\colon & (T-z)^{-1} u \text{ is analytic in } \mathbb{D}, \\ & \left\langle (T-z)^{-1} u, v\right\rangle\big|_{\mathbb{D}} \in H^2 \text{ for all } v \in H \end{array} \right\},$$

$$\widetilde{H^w_{-}}(T) \stackrel{def}{=} \left\{ \begin{array}{ll} u \in H\colon & (T-z)^{-1} u \text{ is analytic in } \mathbb{C}\setminus\overline{\mathbb{D}}, \\ & \left\langle (I-zT)^{-1} u, v\right\rangle\big|_{\mathbb{D}} \in H^2 \text{ for all } v \in H \end{array} \right\}.$$

Let $D_T \stackrel{def}{=} |I - T^*T|^{1/2}$. The subspace

$$H_{ac}(T) \stackrel{def}{=} Clos\, \widetilde{H_{ac}}(T),$$

$$\widetilde{H_{ac}}(T) \stackrel{def}{=} \left\{ \begin{array}{lll} u \in H: & (i) & (T-z)^{-1} u \text{ is analytic in } \mathbb{C}\setminus\mathbb{T}, \\ & (ii) & D_T\,(T-z)^{-1} u\big|_{\mathbb{D}} \in \mathbf{H}^2, \\ & (iii) & D_T\,(I-zT)^{-1} u\big|_{\mathbb{D}} \in \mathbf{H}^2 \end{array} \right\}$$

is called the (strong) a. c. subspace of the operator T. Elements of linear sets $\widetilde{H^w_{ac}}(T)$ and $\widetilde{H_{ac}}(T)$ are called weak and strong smooth vectors, respectively.

An analog of the proposition above holds for the strong smooth vectors of T [3].

Corollary 2. $H_{ac}^w(T) \supset H_{ac}(T)$.

We now proceed to construct the required examples, first for perturbations of unitary operators. Given a selfadjoint operator D, define $\lambda_j(D)$ to be the eigenvalues of D enumerated in the modulus decreasing order. Let $\{\pi_n\}$, $\pi_n > 0$, be a monotone decreasing sequence.

Theorem 3. *There exists a bounded operator T obeying the following conditions,*

i) $H_{ac}^w(T) = H$,
ii) $H_{ac}(T) = \{0\}$,
iii) $I - T^*T \in \mathbf{S}^p$ *for all* $p > 1$,
iv) T *is similar to a unitary operator.*

Moreover, for any sequence $\{\pi_n\} \notin l^1$ there exists an operator T satisfying the conditions above with iii) replaced by

iii') $|\lambda_n(I - T^*T)| \le \pi_n$.

This theorem is optimal in the sense that the subspaces H_{ac}^w and H_{ac} are known [5, Proposition 4.10 and Theorem C] to coincide[1] if $I - T^*T \in \mathbf{S}^1$, provided that $\mathbb{D} \not\subset \sigma_{ess}(T)$. In the terminology of [9], the theorem says that no condition of the form $I - T^*T \in \mathbf{S}_\pi$ where $\mathbf{S}_\pi$ is a symmetrically-normed ideal of compact operators containing $\mathbf{S}^1$ properly, guarantees the coincidence of H_{ac}^w and H_{ac}.

Notice that the similarity of operators respects weak a. c. subspace, therefore the conditions *i)* and *iv)* combined are equivalent to saying that T is similar to an a. c. unitary operator.

Proof. Let $H = \ell^2(\mathbb{Z})$. We shall construct a sequence $\{\rho_n\}_{n=-\infty}^{+\infty}$ of positive numbers such that the weighted bilateral shift operator T defined by

$$Te_j = \rho_{j-1} e_{j-1}, j \in \mathbb{Z},$$

has the required properties. Assume that

$$\sum_j |\rho_j - 1|^p < \infty \tag{$*$}$$

for any $p > 1$. We are going to need the explicit formula for the resolvent of T,

$$\left((T-\lambda)^{-1} f\right)_m = \begin{cases} \displaystyle\sum_{k<m} f_k \frac{\lambda^{m-k-1}}{\prod_k^{m-1} \rho_j}, & |\lambda| < 1, \\ \displaystyle -\sum_{k \ge m} f_k \frac{\prod_m^{k-1} \rho_j}{\lambda^{k-m+1}}, & |\lambda| > 1. \end{cases}$$

[1]The result in [5] is more general pertaining to operators with spectrum on a curve. In the situation under consideration it is given in the unpublished thesis [4].

Proceeding, let us check that the following implication holds

$$D_T\,(T-z)^{-1}\,u\big|_{\mathbb{D}} \in \mathbf{H}^2 \Longrightarrow \sum_{n>0} \frac{|1-\rho_n^2|}{\prod_0^{n-1}\rho_j^2} < \infty, \tag{**}$$

provided that $u \neq 0$. This is done by a direct computation. In the situation under consideration

$$D_T = \operatorname{diag}\{\left|1-\rho_n^2\right|^{1/2}\}.$$

We have:

$$\begin{aligned}\int_{-\pi}^{\pi} \left\|D_T\,(T-z)^{-1}\,f\right\|^2 d\theta &= \sum_n \left|1-\rho_n^2\right| \int_{-\pi}^{\pi} \left|\sum_{k<n} f_k \frac{z^{n-k-1}}{\prod_k^{n-1}\rho_j}\right|^2 d\theta \\ &= \sum_n \left|1-\rho_n^2\right| \sum_{k<n} r^{2(n-k-1)} \frac{|f_k|^2}{\prod_k^{n-1}\rho_j^2}.\end{aligned}$$

Thus, the function $D_T\,(T-z)^{-1}f$ is in $\mathbf{H}^2$ if, and only if, the quantity

$$\sum_n \left|1-\rho_n^2\right| \sum_{k<n} \frac{|f_k|^2}{\prod_k^{n-1}\rho_j^2} = \sum_k \left(\sum_{n>k} \frac{|1-\rho_n^2|}{\prod_k^{n-1}\rho_j^2}\right) |f_k|^2$$

is finite. This means that the sum in parentheses in the right hand side must be finite for some k. Since this sum, obviously, converges or diverges for all k simultaneously, the implication (**) is established.

The existence of an operator T enjoying the properties $i)$ — $iv)$ will be proved if we construct an example of the sequence $\{\rho_j\}$ such that T is similar to an a. c. unitary operator, the sum in (**) diverges, and condition (*) is satisfied. Let $a_j = 1-1/j$ and let

$$\begin{aligned}\rho_j &= 1, j \leq 1, \\ \rho_{2j} &= a_j, j \geq 1, \\ \rho_{2j+1} &= a_j^{-1}, j \geq 1.\end{aligned}$$

With this choice, (*) and the divergence of the sum in (**) are straightforward. Then, define

$$\begin{aligned}w_{2j+1} &= a_j^{-1}, j \geq 1; \\ w_j &= 1 \quad \text{otherwise}.\end{aligned}$$

The diagonal operator $W = \operatorname{diag}\{w_j\}$, defined by the sequence $\{w_j\}$ in H, is obviously bounded, boundedly invertible, and it is easy to check that $W^{-1}TW$ is the unitary operator of (non-weighted) shift in H.

To verify the second assertion of the theorem one can assume without loss of generality that $\pi_{2j+1} = \pi_{2j}$. It is then enough to take $\{a_j\}$ to be any sequence of positive numbers such that $a_j \to 1$, $|1-a_j| \leq \pi_j/2$, but $\sum |1-a_j| = \infty$, in the construction above. $\square$

Remark. *Theorem 3 shows that the linear resolvent growth condition*

$$\sup_{z\notin\mathbb{T}} \left(\left|1-|z|\right| \left\|(T-z)^{-1}\right\|\right) < \infty, \tag{3}$$

and even the combination of it and the property iii), do not imply the coincidence of H_{ac} and H_{ac}^w. Also, it shows that in general similarity of operators does not respect the strong a. c. subspace.

We now turn to perturbations of selfadjoint operators. In principle, the corresponding example can be obtained by the Caley transform. We prefer to give a direct construction based on the same idea, because the Caley transform of the operator in theorem 3 does not belong to the class of operators for which we have defined the strong a. c. subspace. This is not a real hitch, however, for the Caley transform will indeed be the example required if we use the general definition of $H_{ac}(L)$ from [7] mentioned above.

Let $H = L^2(\mathbb{R})$, and $q(x)$ be a bounded function on $\mathbb{R}$ satisfying[2]

$$\textstyle\sum_n \left(\int_n^{n+1} |q|^2\right)^{p/2} < \infty \text{ for all } p > 1,$$

$$q \notin L^1,$$

$$q \text{ is conditionally integrable.}$$

Let L be the operator in H defined by the differential expression

$$L = i\frac{d}{dx} + iq(x)$$

on its natural domain. Notice that the operator L is similar to the operator $A = i\frac{d}{dx}$:

$$L = WAW^{-1},$$

where W is the operator of multiplication by the function $\exp\left(-\int_{-\infty}^x q\right)$.

Theorem 4. *The operator L obeys the following conditions,*

i) $H_{ac}^w(L) = H$,
ii) $H_{ac}(L) = \{0\}$,
iii) $(L-z)^{-1} - (A-z)^{-1} \in \mathbf{S}^p$ *for all* $p > 1$, $\operatorname{Im} z \neq 0$,
iv) *L is similar to a selfadjoint operator.*

Proof. Since A is an a. c. selfadjoint operator, $i)$ and $iv)$ are immediate. Let $V = \operatorname{Im} L$. Suppose that u is a strong smooth vector of L. Since W commutes with the multiplication by a function, this is equivalent to saying that the restrictions of the function

$$\varphi(z) = |V|^{1/2} (A-z)^{-1} g, \; g = Wu,$$

belong to $\mathbf{H}^2_\pm$ in the respective halfplanes. In turn, the latter is equivalent to the condition

$$\int_{\mathbb{R}} \left\| |V|^{1/2} e^{itA} g \right\|^2 dt < \infty$$

[2] $q(x) = \sin x/x$ is the simplest example.

by the Parseval equality for the vector Fourier transform. We have

$$\int_{\mathbb{R}} \left\| |V|^{1/2} e^{itA} g \right\|^2 dt = \int |q(x)| \, |g(x-t)|^2 \, dx \, dt = \|g\|^2 \int |q(x)| dx = \infty$$

if $g \neq 0$. This proves *ii*). The assertion *iii*) is a corollary of the following result from [10],

For any δ, $1 < \delta < 2$, and any functions f, g satisfying

$$\sum_n \left(\int_n^{n+1} |f|^2 \right)^{\delta/2} < \infty, \; \sum_n \left(\int_n^{n+1} |f|^2 \right)^{\delta/2} < \infty,$$

the operator T in H defined by

$$(Tu)(x) = \int_{\mathbb{R}} f(x) e^{ixy} g(y) u(y) dy$$

belongs to $\mathbf{S}^{\delta}$.

Applied to $f = q$ and $g(y) = (y-z)^{-1}$, $\operatorname{Im} z \neq 0$, this result shows that the operator $V(A-z)^{-1} \in \mathbf{S}^p$, $p > 1$, which implies *iii*). $\square$

§2. Let T be a bounded operator such that $\sigma(T) \subset \mathbb{T}$.

Definition 4. *The closure of the linear set of vectors $u \in H$ such that for all $v \in H$ the nontangential limits*

$$f^{\pm}_{u,v}(z) = \lim_{\substack{w \to z \\ |w|^{\pm 1} \in \mathbb{D}}} \left\langle (T-w)^{-1} u, v \right\rangle$$

exist and coincide for a.e. $z \in \mathbb{T}$, is called the singular subspace of the operator T. It is denoted by $H_s(T)$.

As discussed in the Introduction, the duality problem [5] is the question whether the equality

$$(H^w_{ac}(T))^{\perp} = H_s(T^*) \tag{4}$$

holds.

Proposition [3, Proposition 6.7]**.** *If $I - T^*T \in \mathbf{S}^1$, then (4) is satisfied* [3].

More generally, (4) is known to hold if the characteristic function has weak boundary values a.e. [7]. We are now going to construct an example where this property fails. In the notation of definition 3 let

$$CN(T) \stackrel{def}{=} Clos\left(\widetilde{H^w_+}(T) \vee \widetilde{H^w_-}(T) \right).$$

Let $\{\rho_n\}$, $n \geq 0$, be a monotone sequence of positive numbers tending to 0, and R be an operator in $l^2(\mathbb{Z})$ defined by $Re_n = \rho_{|n|} e_n$, $n \in \mathbb{Z}$. Let $H =$

[3]In fact, the cited proposition in [3] establishes (4) with the strong a. c. subspace in the place of $H^w_{ac}(T)$. The latter and the former coincide when $I - T^*T \in \mathbf{S}^1$.

$l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$, and let U be the operator of right shift in $l^2(\mathbb{Z})$, $Ue_n = e_{n+1}$. Define an operator T in H by

$$T = \begin{pmatrix} U & R \\ 0 & U \end{pmatrix}.$$

Obviously, $\sigma(T) = \mathbb{T}$.

Theorem 5. *Let $\{\rho_j\} \notin l^1$. Then the operator T obeys the following conditions,*

i) $CN(T) \neq H$,
ii) $H_s(T^*) = \{0\}$,
iii) $T = T_0 + S$, *where T_0 is unitary and S is an operator whose singular numbers, $\mu_n(S)$, satisfy $\mu_n(S) \leq \rho_{[n/2]}$.*

Proof. The assertion (iii) is obvious (in fact, $\mu_{2n}(S) = \mu_{2n+1}(S) = \rho_n$ for $n \geq 1$). Then, an elementary calculation gives for any $f, g \in H$ and $\lambda \notin \mathbb{T}$,

$$\begin{aligned}\left\langle (T^* - \lambda)^{-1} f, g \right\rangle &= \left\langle (U^* - \lambda)^{-1} f_1, g_1 \right\rangle + \\ &\qquad \left\langle (U^* - \lambda)^{-1} f_2 - (U^* - \lambda)^{-1} R\, (U^* - \lambda)^{-1} f_1, g_2 \right\rangle.\end{aligned}$$

Here

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix};\ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Let f be from the dense set in the definition of $H_s(T^*)$. Considering the g's with $g_2 = 0$ and arbitrary g_1 we conclude that $f_1 = 0$, since U is an absolutely continuous unitary operator. Then, by the same reason, considering the g's with $g_1 = 0$ we obtain that $f_2 = 0$, that is, $H_s(T^*)$ is trivial. Also, the absolute continuity of U implies that $CN(T) \supset \binom{l^2(\mathbb{Z})}{0}$. Let us show that in fact $CN(T) = \binom{l^2(\mathbb{Z})}{0}$. Actually, we shall show that if for a $u = \binom{u_1}{u_2} \in H$ the function

(5)
$$\left\langle (T - \lambda)^{-1} u, \begin{pmatrix} e_j \\ 0 \end{pmatrix} \right\rangle = \left\langle (U - \lambda)^{-1} u_1, e_j \right\rangle - \left\langle (U - \lambda)^{-1} R\, (U - \lambda)^{-1} u_2, e_j \right\rangle$$

is in H^2 for all j, and the H^2-norm of it is bounded above in j, then $u_2 = 0$. Since for a $u \in \widetilde{H^w_+}$ this norm is indeed bounded in j, this is going to imply that $\widetilde{H^w_+} \subset \binom{l^2(\mathbb{Z})}{0}$.

Taking into account that

$$\left\| \left\langle (U - \cdot)^{-1} u_1, e_j \right\rangle \right\|_{H^2}^2 = \sum_1^\infty |u_{1,k+j}|^2 \leq \|u\|^2,$$

we only have to check that the H^2-norm of the second term in the right hand side in (5) is unbounded as $j \to \infty$ if $u_2 \neq 0$. Indeed, a straightforward

calculation gives that for $\lambda \in \mathbb{D}$

$$\left\langle (U-\lambda)^{-1} R\,(U-\lambda)^{-1} u_2, e_j \right\rangle = \sum_{s=0}^{\infty} \lambda^s u_{2,s+j+2} \sum_{m=1}^{s+1} \rho_{|m+j|}.$$

Thus, the H^2 -norm of the left hand side is

$$\sum_{s=0}^{\infty} \left|u_{2,s+j+2}\right|^2 \left| \sum_{m=j+1}^{j+s+1} \rho_{|m|} \right|^2 .$$

Suppose that $u_{2,r} \neq 0$ for some r. Then for j negative enough this norm is bounded below by $\left|u_{2,r}\right|^2 \left| \sum_{m=j+1}^{r-1} \rho_{|m|} \right|^2$ which goes to infinity when $j \to -\infty$ by the assumption about ρ_j. The inclusion $\widetilde{H_-^w} \subset \binom{l^2(\mathbb{Z})}{0}$ is checked similarly. □

In the example constructed the subspaces $CN(T)$ and $H^w_{ac}(T)$ coincide, thus showing that even the condition $CN(T)^\perp \subset H_s(T^*)$, weaker than (4) in general, can fail for non-trace class perturbations. It would be interesting to see if the linear resolvent growth condition (3), violated in the example above, implies the inclusion $CN(T)^\perp \subset H_s(T^*)$.

§3. We now turn to the case $p \neq 2$.

Lemma 6. $\overline{(L-z_0)^{-1} H^{w,p_1}_{ac}} = H^{w,p_1}_{ac} \subset H^{w,p_2}_{ac}$ *for all* $z_0 \in \rho(L)$ *and* $1 \leq p_2 \leq p_1 \leq 2$ *save for* $p_1 = 1$.

Proof. An application of the Hölder inequality to the resolvent identity

$$f_{(L-z_0)^{-1}u,v}(z) = \frac{1}{z-z_0}\left(f_{u,v}(z) - f_{u,v}(z_0)\right)$$

shows that

$$(L-z_0)^{-1}\, \widetilde{H^{w,p_1}_{ac}} \subset \widetilde{H^{w,p_2}_{ac}}$$

for all $z_0 \in \rho(L)$. Since the linear set in the left hand side is independent of the choice of $z_0 \in \rho(L)$, the asymptotics (2) implies that the set is dense in H^{w,p_1}_{ac}. □

It is also easy to check that

$$\left\|(L-z)^{-1} u\right\| \leq C_u \left|\operatorname{Im} z\right|^{-1/p} \tag{6}$$

for any $u \in \widetilde{H^{w,p}_{ac}}$, $1 < p \leq 2$.

Theorem 7. *If L is dissipative ($V \geq 0$) and $1 < p \leq 2$, then* $H^{w,p}_{ac}(L) = H^w_{ac}(L)$.

In the earlier note [6] we proved that if L is dissipative then $H_{ac}(L) = H^w_{ac}(L)$.

Proof. Since the inclusion $H_{ac}^{w} \subset H_{ac}^{w,p}$ is contained in lemma 6, it remains to check that $H_{ac}^{w,p} \subset H_{ac}$. We do so by proving that any vector u from a dense subset in $H_{ac}^{w,p}$ is a strong smooth vector. Recall [2] that for a dissipative operator L the restriction of the function $V^{1/2}\left(L-z\right)^{-1}w$ to $\mathbb{C}_{-}$ belongs to $\mathbf{H}_{-}^{2}$ for all $w \in H$. Hence, we only have to verify that $\left(V^{1/2}\left(L-z\right)^{-1}u\right)_{+} \in \mathbf{H}_{+}^{2}$ for all u from the dense subset.

Let $u \in \left(L+i\right)^{-2}\widetilde{H_{ac}^{w,p}}$. Taking into account (6) one easily checks that the function $\left(L-\cdot-i\varepsilon\right)^{-1}u \in L^{2}(\mathbb{R},H)$ for any $\varepsilon > 0$. We are first going to show that

$$\sup_{\varepsilon>0}\left(\varepsilon\int_{\mathbb{R}}\left\|\left(L-k-i\varepsilon\right)^{-1}u\right\|^{2}dk\right) < \infty. \tag{7}$$

For $\varepsilon > 0$ and $t < 0$ we define

$$u(t) = -\frac{1}{2\pi i}\lim_{N\to\infty}\int_{-N}^{N}e^{i(k+i\varepsilon)t}\left(L-k-i\varepsilon\right)^{-1}u\,dk. \tag{8}$$

Then

1°. The limit in (8) exists for all $t < 0$, and is independent of $\varepsilon > 0$.

2°. $\sup_{t<0}\|u(t)\| < \infty$.

The assertion 1° follows immediately from the possibility to rewrite (8) in the form $(\lambda = k + i\varepsilon)$

$$u(t) = -\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{e^{i\lambda t}}{\left(\lambda+i\right)^{2}}\left(L-\lambda\right)^{-1}u_{2}\,dk - e^{t}(it\,u_{1}+u)$$

where $u_1 = (L+i)u$, $u_2 = (L+i)^2u$.

Let us establish 2°. For any $v \in H$ the scalar product $\langle u(t), v\rangle$ equals to

$$-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{e^{i\lambda t}}{\left(\lambda+i\right)^{2}}f_{u_2,v}(\lambda)\,dk + r_t$$

where $|r_t| \leq C\|v\|$, the bound being uniform in $t < 0$. Since $f_{u_2,v} \in H^p$, one can pass to the limit $\varepsilon \to 0$ in this integral, and use the Hölder inequality to estimate the modulus of it by

$$C\left\|f_{u_2,v}\right\|_{H^p} \leq C\|v\|$$

with a constant C independent of t and v. This implies 2°.

Let $\mathcal{F}$ stands for the conjugate Fourier transform in $L^2(\mathbb{R},H)$. Then (8) means that the restriction of the function

$$\Psi(t) = -\frac{1}{i\sqrt{2\pi}}\mathcal{F}\left[\left(L-\cdot-i\varepsilon\right)^{-1}u\right]$$

to $t < 0$ coincides with $e^{\varepsilon t}u(t)$. On the other hand, $\Psi(t) = 0$ for $t > 0$ by the Paley - Wiener theorem. Applying the Parseval equality and taking into account the property 2°, we find

$$\int_{\mathbb{R}}\left\|\left(L-k-i\varepsilon\right)^{-1}u\right\|^{2}dk = C\int_{-\infty}^{0}e^{2\varepsilon t}\left\|u(t)\right\|^{2}\,dt \leq C\varepsilon^{-1}.$$

The estimate (7) is proved.

We are now going to use the following easily verified identity valid for all $\lambda \in \rho(L)$ ($\varepsilon = \operatorname{Im}\lambda$),

$$\left\|V^{1/2}(L-\lambda)^{-1}u\right\|^2 = \varepsilon\left\|(L-\lambda)^{-1}u\right\|^2 - \operatorname{Im} f_{u,u}(\lambda).$$

The second term in the right hand side can be rewritten in the form

$$\operatorname{Im}\left[\frac{1}{\lambda+i}\left(f_{u_1,u}(\lambda) - f_{u_1,u}(-i)\right)\right]$$

which implies that the term is conditionally integrable in $\operatorname{Re}\lambda$ over the real line, and the integrals are uniformly bounded in ε. Together with (7), this shows that $u \in \widetilde{H_{ac}}$. It remains to notice that $(L+i)^{-2}\,\widetilde{H_{ac}^{w,p}}$ is dense in $H_{ac}^{w,p}$. □

Remark. *In a similar way, the subspaces $H_{ac}^{w,p}(T)$ can be defined for perturbations of unitary operators. These subspaces can be shown to coincide with $H_{ac}(T)$ for all p, $1 \le p \le 2$. The subspaces $H_{ac}^{w,p}(T)$ can also be defined for $0 < p < 1$ but they coincide with H for any unitary operator T, rendering the definition meaningless. The reason is that the Cauchy transform of any finite measure is in $H^p(\mathbb{D})$ for $0 < p < 1$. In the real line context, the definition of the subspace for $p = 1$ requires a regularization at infinity.*

Appendix

Proposition 8. *Let L be a selfadjoint operator and let $\mathcal{H}_{ac}(L)$ be its a. c. subspace defined via the spectral theorem. Then $H_{ac}^{w,p}(L) = \mathcal{H}_{ac}(L)$ for all $p \in (1,2]$.*

Proof. Let $d\mu_{u,v}(t)$ be the matrix element of the spectral measure of L on vectors $u, v \in H$. Then

$$f_{u,v}(z) = \int_{\mathbb{R}} \frac{1}{t-z}\, d\mu_{u,v}(t)$$

for all $u, v \in H$. Let $u = (L-z_0)^{-1} w$ with a $w \in \widetilde{H_{ac}^{w,p}}$ and $z_0 \in \rho(L)$, then $f_{u,v}$ is represented as the Cauchy transform of its boundary values,

$$f_{u,v}(z) = \int_{\mathbb{R}} \frac{1}{(t-z)(t-z_0)}\, f_{w,v}(t)dt.$$

Notice that $(t-z_0)^{-1} f_{w,v}(t)\,dt$ is a finite Borel measure. Comparing the two representations and using the Riesz brothers theorem, one concludes that the measure

$$d\mu_{u,v} - (t-z_0)^{-1} f_{w,v}(t)dt$$

is a. c. for all v. Hence $d\mu_{u,v}$ is a. c. as well. Since the set of such u's is dense in $H_{ac}^{w,p}$, the inclusion $H_{ac}^{w,p} \subset \mathcal{H}_{ac}$ follows. The inclusion $u \in H_{ac}^{w,p}$ is obvious for any $u \in \mathcal{H}_{ac}$ satisfying $d\mu_{u,u}/dt \in L^\infty(\mathbb{R})$. Since the set of such u's is dense in $u \in \mathcal{H}_{ac}$, it implies that $\mathcal{H}_{ac} \subset H_{ac}^{w,p}$. □

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